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Boundary K-matrix, elliptic Dunkl operator and quantum many-body systems

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Abstract. The infinite-dimensional representation of the boundary K-operator (the solution of the reflection equation) is considered. The trigonometric K-matrix is studied as a degenerate case of the elliptic operator. A method to construct the elliptic Dunkl operator is proposed, and the relationship with the quantum many-body problem is also discussed.

1. Introduction

Solutions of the Yang–Baxter equation have been widely studied. The algebraic structure has been revealed as the quantum group [1]. Among solutions, the infinite-dimensional representation of the R-matrix was recently given [2, 3]; the R-matrix is viewed as an operator acting on a two-variable function space. The elliptic solution proposed by Shibukawa and Ueno has become a useful operator both to construct the finite-dimensional representation of the R-matrix and to give the mutually commuting differential operators. In this paper, we shall investigate the structures and applications of the R-operator and boundary K-operator.

Throughout this paper we use the doubly quasi-periodic function $\sigma_\mu(z) \equiv \sigma_\mu(z, \tau)$,

$$\sigma_\mu(z+1) = \sigma_\mu(z) \qquad \sigma_\mu(z+\tau) = \mathrm{e}^{2\pi \mathrm{i}\mu}\sigma_\mu(z) \tag{1.1}$$

where τ is an arbitrary number, satisfying $\tau \in \mathbb{C}$ and $\Im\tau > 0$. The function $\sigma_\mu(z)$ only has simple poles on the lattice $\mathbb{Z} + \tau\mathbb{Z}$, and the residue at the origin is one. Note that the function $\sigma_\mu(z)$ can be written explicitly as

$$\sigma_\mu(z) = \frac{\vartheta_1(z-\mu)\vartheta_1'(0)}{\vartheta_1(z)\vartheta_1(-\mu)} \tag{1.2}$$

where $\vartheta_1(z) \equiv \vartheta_1(z, \tau)$ is the Jacobi theta function,

$$\vartheta_1(z,\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \exp\bigl(\mathrm{i}\pi n^2\tau + 2\pi \mathrm{i} n\bigl(z+\tfrac{1}{2}\bigr)\bigr)\,. \tag{1.3}$$

The elliptic function $\sigma_\mu(z)$ has the following properties (see, for example, [4]):

$$\sigma_\mu(z) = -\sigma_{-\mu}(-z) \tag{1.4a}$$

$$\sigma_\mu(z) = -\sigma_z(\mu) \tag{1.4b}$$

$$\sigma_\lambda(z)\sigma_{\lambda+\mu}(w) + \sigma_{\lambda+\mu}(z+w)\sigma_\mu(-z) - \sigma_\mu(w)\sigma_\lambda(z+w) = 0 \tag{1.4c}$$

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$$\sigma_\lambda(z)\sigma_\mu(z) = \sigma_{\lambda+\mu}(z)(\zeta(z) - \zeta(\lambda) - \zeta(\mu) - \zeta(z-\lambda-\mu)) \tag{1.4d}$$

$$\sigma_\mu(z)\sigma_{-\mu}(z) = \wp(z) - \wp(\mu) \tag{1.4e}$$

$$\lim_{\mu\to 0} \frac{\mathrm{d}}{\mathrm{dz}} \sigma_\mu(z) = -\wp(z) - 2\eta_1 \,. \tag{1.4f}$$

We denoted $\wp(z)$ and $\zeta(z)$ as the Weierstrass $\wp$-function and ζ-function, respectively; $\wp(z) = -\zeta'(z)$.

In terms of the elliptic function $\sigma_\mu(z)$, Shibukawa and Ueno introduced the elliptic R-operator,

$$R(\xi) f(z_1, z_2) = \sigma_\mu(z_{12}) f(z_1, z_2) - \sigma_\xi(z_{12}) f(z_2, z_1) \,. \tag{1.5}$$

Here and hereafter we use the conventional notations, e.g. $z_{12} \equiv z_1 - z_2$. Parameter ξ is called the spectral parameter. The elliptic R-operator defined above satisfies the Yang–Baxter equation (YBE) (see figure 1),

$$R^{12}(\xi_{12}) R^{13}(\xi_{13}) R^{23}(\xi_{23}) = R^{23}(\xi_{23}) R^{13}(\xi_{13}) R^{12}(\xi_{12}) \,. \tag{1.6}$$

The operator R^{jk} acts on a function of N variables by viewing it as a function of the jth and kth variable. Note that we obtain the rational and trigonometric R-operators as degenerate cases of elliptic operator (1.5):

$$\sigma_\mu(z) \to \begin{cases} \pi \cot(\pi z) - \pi \cot(\pi \mu) & \text{trigonometric} \\ z^{-1} - \mu^{-1} & \text{rational}\,. \end{cases} \tag{1.7}$$

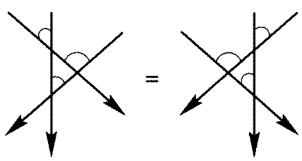

Figure 1. Yang–Baxter equation.

Besides the R-matrix as a solution of YBE, the boundary K-matrix was introduced to solve the spin system with an open boundary using the technique of quantum inverse scattering [5, 6]; algebraically the K-matrix is defined as a solution of the so-called reflection equation (RE, or boundary YBE). As in the case of the R-matrix, we can also regard the K-matrix as an operator acting on a functional space [7]. We define boundary K-operators which act on a space of functions of single variable as

$$K^{\mathrm{I}}(\xi) f(z) = \sigma_{2\xi}(z) f(z) - \sigma_{2\nu}(z) f(-z) \tag{1.8a}$$

$$K^{\mathrm{II}}(\xi) f(z) = \sigma_\xi(2z) f(z) - \sigma_\nu(2z) f(-z) \,. \tag{1.8b}$$

These two K-operators satisfy RE (see figure 2),

$$\begin{aligned} &R^{12}(\xi_{12})(K(\xi_1)\otimes 1) R^{21}(\xi_1+\xi_2)(1\otimes K(\xi_2)) \\ &\qquad = (1\otimes K(\xi_2)) R^{12}(\xi_1+\xi_2)(K(\xi_1)\otimes 1) R^{21}(\xi_{12}) \end{aligned} \tag{1.9}$$

where $R(\xi)$ means Shibukawa–Ueno's elliptic operator (1.5) satisfying YBE.

The validity of RE is proved directly by use of identities for the elliptic function $\sigma_\mu(z)$ (equation (1.4)). It is remarkable that we have two solutions, type I and type II. The roles of these two K-operators may be regarded as a 'reflection' associated with the classical root systems of B and C type. This will be clarified in this paper.

The rest of this paper is organized as follows. In the first part, we consider the trigonometric representation of the R- and K-matrices. The elliptic R-matrix, which is

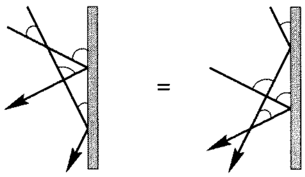

Figure 2. Reflection equation.

called Belavin's completely $\mathbb{Z}$-symmetric R-matrix [8], can be obtained using the 'gauge transformation' of the elliptic R-operator [9]. The Belavin R-matrix reduces in the simple case to the Baxter eight-vertex model. The boundary K-matrix associated with Belavin's R-matrix can also be computed from the 'gauge-transformed' elliptic K-operator [10]. These modified operators are essentially the same as those used in the definition of the elliptic Dunkl operators. In section 2, we construct the finite-dimensional representation of the R- and K-matrices for the trigonometric case, which can be obtained as a degenerate case of the Belavin elliptic matrix.

In the second part of this paper, we shall construct the elliptic Dunkl operator based on YBE and RE. The Dunkl operator was first introduced as a set of integrable differential-difference operators associated with the root system [11]. The Dunkl operator plays an important role in studying the algebraic structures of certain one-dimensional integrable many-body problems with inverse-square exchange (Calogero–Sutherland–Moser model) [12–16]. Though the Dunkl operator was originally defined as a 'rational' operator, one can generalize operators into trigonometric [17] and elliptic [18, 19] cases. Both the rational and trigonometric operators are regarded as degenerate cases of the elliptic Dunkl operators. In section 3 we give a simple method to construct the elliptic Dunkl operators associated with the classical root systems by using the infinite-dimensional representation of the R- and K-operators. This method is a generalization of [18]. We define a set of mutually commuting elliptic difference operator (quantum Knizhnik–Zamolodchikov–Bernard difference operator), and show that the elliptic Dunkl operator appears as a quasi-classical limit. Also shown is that these elliptic Dunkl operators constitute mutually commuting Hamiltonian sets of Calogero–Sutherland–Moser systems associated with the classical root systems.

2. Finite-dimensional representation

We construct the 'finite-dimensional' representation of the R- and K-matrix by restricting the functional space to a finite-dimensional space. In order to construct Belavin's $\mathbb{Z}$-symmetric R-matrix and its boundary K-matrix whose elements are elliptic, one should use the gauge-transformed elliptic R- and K-operators [9, 10]. We note that these operators are essentially the same as the modified R- and K-operators used in the next section. In this section, we treat the trigonometric case as a degenerate case of the elliptic function (1.7), and discuss the algebraic structure for a simple case.

We choose bases $f_a(z)$ of the finite-dimensional functional space as

$$f_a(z) = \exp(2\pi \mathrm{i} a z) \,. \tag{2.1}$$

For an n-dimensional functional space V_n, the index a takes a value in $a \in \{-\frac{n-1}{2}, -\frac{n-1}{2}+1, \ldots, \frac{n-1}{2}\}$. For these bases, the action of the R-operator is written explicitly as an $n^2 \times n^2$-matrix. We define the $R_{(n)}$-matrix as

$$R_{(n)}(\theta) = \frac{1}{2\mathrm{i}}(\theta^{1/2} - \theta^{-1/2})(q^{1/2} - q^{-1/2}) R(\xi) \tag{2.2}$$

where the spectral parameter and deformation parameter are defined by $\theta = \exp(2\pi \mathrm{i}\xi)$ and $q = \exp(2\pi \mathrm{i}\mu)$, respectively. It is easy to calculate the matrix elements of this $R_{(n)}$-matrix [2],

$$R_{(n)}(\theta) f_a \otimes f_b = \begin{cases} -q^{-1/2}(\theta^{1/2} - \theta^{-1/2}) f_a \otimes f_b + \theta^{1/2}(q^{1/2} - q^{-1/2}) f_b \otimes f_a \\ \quad +(\theta^{1/2} - \theta^{-1/2})(q^{1/2} - q^{-1/2}) \sum_{b<c<a} f_{a+b-c} \otimes f_c \\ \qquad \text{for} \quad a > b \\ (\theta^{-1/2} q^{1/2} - \theta^{1/2} q^{-1/2}) f_a \otimes f_a \qquad \text{for} \quad a = b \\ -q^{1/2}(\theta^{1/2} - \theta^{-1/2}) f_a \otimes f_b + \theta^{-1/2}(q^{1/2} - q^{-1/2}) f_b \otimes f_a \\ \quad -(\theta^{1/2} - \theta^{-1/2})(q^{1/2} - q^{-1/2}) \sum_{a<c<b} f_c \otimes f_{a+b-c} \\ \qquad \text{for} \quad a < b\,. \end{cases} \tag{2.3}$$

This $R_{(n)}$-matrix maps $V_n \otimes V_n$ to $V_n \otimes V_n$.

The trigonometric boundary K-matrices associated with $R_{(n)}$-matrix are also constructed from the infinite-dimensional K-operator (1.8a) in the same manner. We set the finite-dimensional representation of the boundary K-matrices as

$$K^{\mathrm{I}}_{(n)}(\theta) = \frac{1}{2\mathrm{i}}(\theta^{1/2} - \theta^{-1/2})(p^{1/2} - p^{-1/2}) K^{\mathrm{I}}(\xi) \tag{2.4a}$$

$$K^{\mathrm{II}}_{(n)}(\theta) = \frac{1}{2\mathrm{i}}(\theta - \theta^{-1})(p - p^{-1}) K^{\mathrm{II}}(\xi) \tag{2.4b}$$

with $p = \exp(2\pi \mathrm{i}\nu)$. By definition, we obtain an explicit n-dimensional representation for the reflection K-matrices as follows:

$$K^{\mathrm{I}}_{(n)}(\theta) f_a = \begin{cases} -\theta^{-1/2}(p^{1/2} - p^{-1/2}) f_a + p^{1/2}(\theta^{1/2} - \theta^{-1/2}) f_{-a} \\ \quad +(\theta^{1/2} - \theta^{-1/2})(p^{1/2} - p^{-1/2}) \sum_{c<|a|} f_c & \text{for} \quad a > 0 \\ (\theta^{1/2} p^{-1/2} - \theta^{-1/2} p^{1/2}) f_0 & \text{for} \quad a = 0 \\ -\theta^{1/2}(p^{1/2} - p^{-1/2}) f_a + p^{-1/2}(\theta^{1/2} - \theta^{-1/2}) f_{-a} \\ \quad -(\theta^{1/2} - \theta^{-1/2})(p^{1/2} - p^{-1/2}) \sum_{c<|a|} f_c & \text{for } a < 0 \end{cases} \tag{2.5a}$$

$$K^{\mathrm{II}}_{(n)}(\theta) f_a = \begin{cases} -\theta^{-1}(p - p^{-1}) f_a + p(\theta - \theta^{-1}) f_{-a} \\ \quad +(\theta - \theta^{-1})(p - p^{-1}) \sum_{c<|a|} f_c & \text{for } a > 0 \\ (\theta p^{-1} - \theta^{-1} p) f_0 & \text{for} \quad a = 0 \\ -\theta(p - p^{-1}) f_a + p^{-1}(\theta - \theta^{-1}) f_{-a} \\ \quad -(\theta - \theta^{-1})(p - p^{-1}) \sum_{c<|a|} f_c & \text{for} \quad a < 0\,. \end{cases} \tag{2.5b}$$

By extracting the ‘permutation part’ [2] of these matrices, we obtain the triangular R-matrix of Drinfeld and its reflection K-matrices. We can check directly that both YBE (1.6) and

RE (1.9) are satisfied by the following triangular R- and K-matrices:

$$R_{(n)}(\theta) f_a \otimes f_b = \begin{cases} -q^{-1/2}(\theta^{1/2}-\theta^{-1/2}) f_a \otimes f_b + \theta^{1/2}(q^{1/2}-q^{-1/2}) f_b \otimes f_a & \text{for} \quad a > b \\ (\theta^{-1/2}q^{1/2} - \theta^{1/2}q^{-1/2}) f_a \otimes f_a, & \text{for} \quad a = b \\ -q^{1/2}(\theta^{1/2}-\theta^{-1/2}) f_a \otimes f_b + \theta^{-1/2}(q^{1/2}-q^{-1/2}) f_b \otimes f_a & \text{for} \quad a < b \end{cases} \tag{2.6}$$

$$K^{\mathrm{I}}_{(n)}(\theta) f_a = \begin{cases} -\theta^{-1/2}(p^{1/2} - p^{-1/2}) f_a + p^{1/2}(\theta^{1/2} - \theta^{-1/2}) f_{-a} & \text{for} \quad a > 0 \\ (\theta^{1/2}p^{-1/2} - \theta^{-1/2}p^{1/2}) f_0 & \text{for} \quad a = 0 \\ -\theta^{1/2}(p^{1/2} - p^{-1/2}) f_a + p^{-1/2}(\theta^{1/2} - \theta^{-1/2}) f_{-a} & \text{for} \quad a < 0 \end{cases} \tag{2.7a}$$

$$K^{\mathrm{II}}_{(n)}(\theta) f_a = \begin{cases} -\theta^{-1}(p - p^{-1}) f_a + p(\theta - \theta^{-1}) f_{-a} & \text{for} \quad a > 0 \\ (\theta p^{-1} - \theta^{-1} p) f_0, & \text{for} \quad a = 0 \\ -\theta(p - p^{-1}) f_a + p^{-1}(\theta - \theta^{-1}) f_{-a} & \text{for} \quad a < 0\ . \end{cases} \tag{2.7b}$$

We consider the algebraic structures of the two-dimensional representation for R- and K-matrices in more detail [20]. In this case, R- and K-matrices can be written as follows:

$$R_{(2)}(\theta) = \theta^{-1/2} \cdot R - \theta^{1/2} \cdot P R^{-1} P \tag{2.8}$$

$$K^{\mathrm{I}}_{(2)}(\theta) = \theta^{1/2} \cdot K_{\mathrm{I}}^{-1} - \theta^{-1/2} \cdot K_{\mathrm{I}} \tag{2.9}$$

$$K^{\mathrm{II}}_{(2)}(\theta) = \theta \cdot K_{\mathrm{II}}^{-1} - \theta^{-1} \cdot K_{\mathrm{II}} \tag{2.10}$$

where constant matrices R and $K_{\mathrm{I,II}}$ are defined as

$$R = \begin{pmatrix} q^{1/2} & & & \\ & q^{1/2} & q^{1/2} - q^{-1/2} & \\ & & q^{-1/2} & \\ & & & q^{1/2} \end{pmatrix} \tag{2.11}$$

$$K_{\mathrm{I}} \equiv K = \begin{pmatrix} 0 & p^{-1/2} \\ p^{1/2} & p^{1/2} - p^{-1/2} \end{pmatrix} \tag{2.12}$$

$$K_{\mathrm{II}} = K \sigma_x K = \begin{pmatrix} 0 & p^{-1} \\ p & p - p^{-1} \end{pmatrix}. \tag{2.13}$$

Matrices P and σ_x denote the permutation matrix and the Pauli spin matrix, respectively. By substituting the definition of the R- and K-matrices into YBE and RE, we get relations among matrices R and K,

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12} \tag{2.14}$$

$$R - P R^{-1} P = (q^{1/2} - q^{-1/2}) P \tag{2.15}$$

$$K - K^{-1} = p^{1/2} - p^{-1/2} \tag{2.16}$$

$$\big(1 \otimes K\big) R^{21} \big(K \otimes 1\big) R^{12} = R^{21} \big(K \otimes 1\big) R^{12} \big(1 \otimes K\big)\,. \tag{2.17}$$

These relations can be taken as the defining relations for the q-deformed affine $SL(2)$ Lie algebra. We call the first identity the constant Yang–Baxter equation. The second and the third equations are called the Hecke relation. Both K- and R-matrices are regarded as representations of the Hecke algebra. The difference between two representations originates from that the R-operator acts on a two-variable functional space while the K-operator acts on a single-variable space. Thus the fourth relation (constant reflection equation) can be viewed as the interrelation between the two representations [21].

3. Elliptic Dunkl operator

We shall define the elliptic Dunkl operators associated with the classical root systems based on the elliptic R- and K-operators [18, 19]. The Dunkl operator is first defined in [11] as a set of integrable differential-difference operators associated with the root systems. These operators become a powerful tool in studying algebraic structures of the one-dimensional integrable many-body problems with inverse-square exchange (Calogero–Sutherland–Moser model, CSM model for short). For the trigonometric Dunkl operators associated with the classical root systems, a method has been recently proposed to systematically define them from the quantum Knizhnik–Zamolodchikov (qKZ) type difference operators [22, 7, 23]. The qKZ difference operators are constructed in terms of the R- and K-operators, and constitute a mutually commuting family. In this section, using the gauge-transformed elliptic R- and K-operators [18, 9, 10], we construct the elliptic Dunkl operators associated with the classical root systems.

For our purpose, we use another boundary operator $\bar{K}(\xi)$. The $\bar{K}$-operator is defined as a solution of the 'conjugate' reflection equation (RE2),

$$\begin{aligned} &R^{12}(\xi_{21})(1 \otimes \bar{K}(\xi_2))R^{21}(\xi_1+\xi_2)(\bar{K}(\xi_1)\otimes 1) \\ &\quad = (\bar{K}(\xi_1)\otimes 1)R^{12}(\xi_1+\xi_2)(1\otimes \bar{K}(\xi_2))R^{21}(\xi_{21})\,. \end{aligned} \tag{3.1}$$

Notice that the $\bar{K}$-operator is related with the K-operator, which is defined as a solution of RE (1.9), as

$$\bar{K}(\xi) = \hat{t}K(\xi)\hat{t} \tag{3.2}$$

where $\hat{t}$ means a reflection operator acting on a single-variable functional space,

$$\hat{t}f(z) = f(-z)\,. \tag{3.3}$$

Actually one can see that RE2 (equation (3.1)) is proved by using relation (3.2) from RE (1.9). As a consequence, we also have a two-conjugate boundary operator, $\bar{K}^{\mathrm{I,II}}(\xi)$, as solutions of RE2.

Based on the method of [18], we introduce the shift operator $T_\kappa(\xi)$,

$$T_\kappa(\xi)f(z) = f\left(z-\frac{\xi}{\kappa}\right). \tag{3.4}$$

Here κ is an arbitrary parameter. With the shift operator $T_\kappa(\xi)$, we define the modified (gauge-transformed) R- and K-operators as

$$R_\kappa(\xi) = (1\otimes T_\kappa(\mu))\cdot R(\xi)\cdot(T_\kappa(-\mu)\otimes 1) \tag{3.5}$$

$$K_\kappa^{\mathrm{I,II}}(\xi) = T_\kappa(-\nu)\cdot K^{\mathrm{I,II}}(\xi)\cdot T_\kappa(\nu)\,. \tag{3.6}$$

The crucial point is that a set of modified operators, $R_\kappa(\xi)$ and $K_\kappa(\xi)$, also satisfies YBE and RE;

$$R_\kappa^{12}(\xi_{12})R_\kappa^{13}(\xi_{13})R_\kappa^{23}(\xi_{23}) = R_\kappa^{23}(\xi_{23})R_\kappa^{13}(\xi_{13})R_\kappa^{12}(\xi_{12}) \tag{3.7}$$

$$\begin{aligned} &R_\kappa^{21}(\xi_{12})(K_\kappa(\xi_1)\otimes 1)R_\kappa^{12}(\xi_1+\xi_2)(1\otimes K_\kappa(\xi_2)) \\ &\quad = (1\otimes K_\kappa(\xi_2))R_\kappa^{21}(\xi_1+\xi_2)(K_\kappa(\xi_1)\otimes 1)R_\kappa^{12}(\xi_{12})\,. \end{aligned} \tag{3.8}$$

Above YBE and RE are directly checked using properties of the shift operators,

$$T_\kappa(\xi)T_\kappa(\eta) = T_\kappa(\xi+\eta) \tag{3.9}$$

$$[R(\xi), T_\kappa(\eta)\otimes T_\kappa(\eta)] = 0\,. \tag{3.10}$$

The first one is trivial. The second identity follows from that operator $R(\xi)$ depends only on the difference of two functional variables. We note the explicit actions of these modified operators;

- R_κ-operator,

$$R_\kappa(\xi) f(z_1, z_2) = \frac{1}{\sigma_\mu(\xi)} \left\{ \sigma_\mu \left(z_{12} + \frac{\mu}{\kappa} \right) f \left(z_1 + \frac{\mu}{\kappa}, z_2 - \frac{\mu}{\kappa} \right) - \sigma_\xi \left(z_{12} + \frac{\mu}{\kappa} \right) f(z_2, z_1) \right\} \tag{3.11}$$

- K_κ-operators,

$$K_\kappa^{\mathrm{I}}(\xi) f(z) = \frac{1}{\sigma_{2\nu}(2\xi)} \left\{ \sigma_{2\xi} \left(z + \frac{\nu}{\kappa} \right) f(z) - \sigma_{2\nu} \left(z + \frac{\nu}{\kappa} \right) f \left(-z - \frac{2\nu}{\kappa} \right) \right\} \tag{3.12a}$$

$$K_\kappa^{\mathrm{II}}(\xi) f(z) = \frac{1}{\sigma_\nu(\xi)} \left\{ \sigma_\xi \left(2z + \frac{2\nu}{\kappa} \right) f(z) - \sigma_\nu \left(2z + \frac{2\nu}{\kappa} \right) f \left(-z - \frac{2\nu}{\kappa} \right) \right\}. \tag{3.12b}$$

Here for our convenience, we have normalized the modified operators to satisfy the unitarity condition

$$R_\kappa^{12}(\xi) R_\kappa^{21}(-\xi) = 1 \tag{3.13}$$

$$K_\kappa(\xi) K_\kappa(-\xi) = 1\,. \tag{3.14}$$

See also that we have the quasi-classical conditions for these operators;

$$\begin{aligned} &R_\kappa(\xi = 0) = \hat{s} \qquad R_\kappa(\xi)|_{\mu=0} = 1 \\ &K_\kappa(\xi = 0) = 1 \qquad K_\kappa(\xi)|_{\nu=0} = \hat{t}\,. \end{aligned} \tag{3.15}$$

Operator $\hat{s}$ denotes an exchange operator acting on a two-variable functional space,

$$\hat{s}_{12} f(z_1, z_2) = f(z_2, z_1)\,. \tag{3.16}$$

Based on these modified operators, we shall show that the elliptic Dunkl operators associated with the classical root systems are defined from the mutually commuting difference operators. First we review the construction of the A-type Dunkl operator [18]. We use a set of difference operators $\{\hat{T}_j | j = 1, \ldots, N\}$ in terms of the modified R-operator as

$$\hat{T}_j = R_\kappa^{jj-1}(\xi_{jj-1}) \ldots R_\kappa^{j1}(\xi_{j1}) \cdot R_\kappa^{jN}(\xi_{jN}) \ldots R_\kappa^{jj+1}(\xi_{jj+1})\,. \tag{3.17}$$

Here R_κ^{jk} signifies the operator on $V^{\otimes N}$, acting as R_κ on the jth and kth spaces and identity on the other space. This operator can be viewed as the 'infinite-dimensional' representation of the inhomogeneous transfer matrix, and has appeared as a qKZ operator [24] or as a form factor equation [25]. The integrability of the operators $\hat{T}_j$,

$$\left[\hat{T}_j, \hat{T}_k\right] = 0 \tag{3.18}$$

follows from YBE (equation (3.7)). Taking parameter μ as an infinitesimal parameter, $\mu \to 0$, we find that the operator $\hat{T}_j$ reduces in a quasi-classical limit into a form,

$$\hat{T}_j = 1 + \frac{\mu}{\kappa} \sum_{k:k\neq j}^{N} \{\partial_j - \partial_k + \kappa \sigma_{\xi_{jk}}(z_{jk}) \hat{s}_{jk} + \kappa(\rho(\xi_{jk}) - \rho(z_{jk}))\} + \mathrm{O}(\mu^2)$$

where we have used notations, $\partial_j = \partial/\partial z_j$ and

$$\rho(z) = \frac{\vartheta_1'(z)}{\vartheta_1(z)}\,.$$

When we assume that the completely translational-invariant functional space (function depends only on differences of any two variables),

$$f(z_1+w, z_2+w, \ldots, z_N+w) = f(z_1, z_2, \ldots, z_N) \qquad w \in \mathbb{C}$$

we have an identity, $\sum_{j=1}^{N} \partial_j = 0$. With this identity, one sees that the difference operator $\hat{T}_j$ is written as

$$\hat{T}_j = 1 + \mu \frac{N}{\kappa} \hat{d}'_j + \mathrm{O}(\mu^2)$$

where we have used the differential-difference operator $\hat{d}'_j$ as

$$\hat{d}'_j = \partial_j + \frac{\kappa}{N} \sum_{k:k\neq j}^{N} \sigma_{\xi_{jk}}(z_{jk}) \hat{s}_{jk} + \frac{\kappa}{N} \sum_{k:k\neq j}^{N} \{\rho(\xi_{jk}) - \rho(z_{jk})\} .$$

It is obvious from the commutativity of $\hat{T}_j$ (3.18) that the operator $\hat{d}'_j$ is integrable; $\left[\hat{d}'_j, \hat{d}'_k\right] = 0$. In order to modify the differential-difference operator $\hat{d}'_j$ into a more useful form, let us define a (translation-invariant) product function,

$$H_a(z) = H_a(z_1, \ldots, z_N) = \prod_{j<k}^{N} (\vartheta_1(z_{jk}))^a$$

and introduce the differential-difference operator $\hat{d}_j$ as a gauge-transformation of $\hat{d}'_j$,

$$\hat{d}_j \equiv (H_a(z))^{-1} \cdot \hat{d}'_j \cdot H_a(z) .$$

By choosing parameter a properly and redefining κ, we get differential-difference operator with elliptic coefficients as a quasi-classical limit of the qKZ operator $\hat{T}_j$,

$$\hat{d}_j = \partial_j + \kappa \sum_{k:k\neq j}^{N} \sigma_{\xi_{jk}}(z_{jk}) \hat{s}_{jk} \tag{3.19}$$

which constitutes a set of mutually commuting differential-difference operators,

$$\left[\hat{d}_j, \hat{d}_k\right] = 0 . \tag{3.20}$$

The operator $\hat{d}_j$ is called the A-type elliptic Dunkl operator. From a set of the commuting operators $\{\hat{d}_j \,|\, j = 1, \ldots, N\}$, one can construct a set of 'Hamiltonians' of the quantum N-body dynamical system,

$$\hat{I}_n^{\mathrm{A}} = \sum_{j=1}^{N} \left(\hat{d}_j\right)^n . \tag{3.21}$$

One of the integrable operators is calculated as

$$\hat{I}_2^{\mathrm{A}} = \sum_{j} \partial_j^2 + \kappa {\sum_{j,k}}' \sigma'_{\xi_{jk}}(z_{jk}) \hat{s}_{jk} + \kappa^2 {\sum_{j,k}}' \{\wp(\xi_{jk}) - \wp(z_{jk})\}$$

where $\sum'$ means that any two indices do not coincide. Subtracting terms of order ξ^{-2} and setting all rapidities to zero ($\xi \to 0$), we obtain the N-body Hamiltonian of the A-type elliptic CSM model with $su(n)$ spin degree of freedom [26],

$$\hat{I}_2^{\mathrm{A}} \to \sum_{j=1}^{N} \partial_j^2 - {\sum_{j,k=1}^{N}}' \wp(z_{jk}) \cdot \left(\kappa \hat{P}_{jk} + \kappa^2\right) . \tag{3.22}$$

Here we use the $su(n)$ spin permutation operator as $\hat{P}_{jk}$, which satisfies

$$\hat{P}_{jk}^2 = 1 \qquad \hat{P}_{jk}\hat{P}_{kl} = \hat{P}_{jl}\hat{P}_{jk} = \hat{P}_{kl}\hat{P}_{jl}\,.$$

Note that we have restricted the functional space to

$$\hat{P}_{jk}\hat{s}_{jk} = 1\,. \tag{3.23}$$

As coefficients of ξ^{-2} are constants, the commutativity between the Hamiltonian of the elliptic CSM model (3.22) and higher operators $\hat{I}_n^{\mathrm{A}}$ is preserved.

In the process of constructing the elliptic A-type Dunkl operator $\hat{d}_j$, we have used only modified R-operator in the qKZ operator (3.17). In the rest of this section, we show that the W-type (W = B, C, BC, D) elliptic Dunkl operator can be constructed using the explicit representation of modified R- and K-operators. As a set of qKZ-type mutually commuting difference operators, we have another family, which may be called the qKZ operator with boundary [5, 7, 27]. This operator is written in terms of modified R- and K-operators as

$$\begin{aligned}\hat{Y}_j = R_\kappa^{jj-1}(\xi_{jj-1})\ldots R_\kappa^{j1}(\xi_{j1})\cdot K_\kappa^{j}(\xi_j)\cdot R_\kappa^{1j}(\xi_1+\xi_j)\ldots R_\kappa^{j-1j}(\xi_{j-1}+\xi_j)\\ \times R_\kappa^{j+1j}(\xi_{j+1}+\xi_j)\ldots R_\kappa^{Nj}(\xi_N+\xi_j)\cdot \bar{K}_\kappa^{j}(\xi_j)\cdot R_\kappa^{jN}(\xi_{jN})\ldots R_\kappa^{jj+1}(\xi_{jj+1})\,.\end{aligned} \tag{3.24}$$

Here K_κ^j and $\bar{K}_\kappa^j$ means the K_κ- and $\bar{K}_\kappa$-operators acting on an N-variable function by viewing it as of the jth variable. The integrability of operators $\hat{Y}_j$,

$$\left[\hat{Y}_j, \hat{Y}_k\right] = 0 \tag{3.25}$$

is proved by use of YBE (3.7) and RE (3.8). Note that, in the description of the qKZ operator (3.24), we can use the boundary K- and $\bar{K}$-operators as both type-I and type-II operators. Thus we have four sets of mutually commuting difference operators. In the following we show that these four sets of operators correspond to those associated with the classical B, C and BC-type root system.

As an example, we use in (3.24) type-I boundary operators K^{I} and $\bar{K}^{\mathrm{I}}$. In this case, the quasi-classical limit of the boundary qKZ operator can be calculated as

$$\begin{aligned}\hat{Y}_j = 1 + \frac{\mu}{\kappa}\Bigg\{&(2(N-1)+\alpha+\bar{\alpha})\partial_j + \kappa\sum_{k:k\neq j}^{N}(\sigma_{\xi_{jk}}(z_{jk})\hat{s}_{jk} + \sigma_{\xi_j+\xi_k}(z_j+z_k)\hat{t}_j\hat{t}_k\hat{s}_{jk})\\ &+\kappa(\alpha+\bar{\alpha})\cdot\sigma_{2\xi_j}(z_j)\hat{t}_j\\ &+\kappa\sum_{k:k\neq j}^{N}(\rho(\xi_{jk}) - \rho(z_{jk}) + \rho(\xi_j+\xi_k) - \rho(z_j+z_k))\\ &+\kappa(\alpha+\bar{\alpha})\cdot(\rho(z_j)+\rho(2\xi_j))\Bigg\} + \mathrm{O}(\mu^2)\,.\end{aligned}$$

Here we have set parameters in operators K_κ and $\bar{K}_\kappa$ as $\nu = \alpha\mu$ and $\bar{\nu} = \bar{\alpha}\mu$, respectively. Following the method in constructing the A-type Dunkl operator, we use a 'B-type' elliptic product function,

$$H_{a,b}^{\mathrm{B}}(z) = \prod_{j<k}^{N}(\vartheta_1(z_{jk})\vartheta_1(z_j+z_k))^a\cdot\prod_{j=1}^{N}(\vartheta_1(z_j))^b\,.$$

When one transforms the operator as

$$\hat{Y}_j \to (H_{a,b}^{\mathrm{B}}(z))^{-1}\cdot\hat{Y}_j\cdot H_{a,b}^{\mathrm{B}}(z)$$

and fixes parameters ($a, b, \alpha, \bar{\alpha}$ and κ) properly, we get the quasi-classical limit of the qKZ operator $\hat{Y}_j$ as

$$\hat{Y}_j = 1 + \frac{\mu}{\kappa} \cdot \hat{y}_j^{\mathrm{B}} + \mathrm{O}(\mu^2),$$

where operator $\hat{y}_j^{\mathrm{B}}$ is defined as

$$\hat{y}_j^{\mathrm{B}} = \partial_j + \kappa_1 \sum_{k:k\neq j}^{N} \{\sigma_{\xi_{jk}}(z_{jk})\hat{s}_{jk} + \sigma_{\xi_j+\xi_k}(z_j + z_k)\hat{t}_j\hat{t}_k\hat{s}_{jk}\} + \kappa_2\sigma_{2\xi_j}(z_j)\hat{t}_j \, . \tag{3.26}$$

Parameters κ_1 and κ_2 denote newly redefined arbitrary parameters. One sees from the integrability of $\hat{Y}_j$ (3.25) that the operator $\hat{y}_j^{\mathrm{B}}$, which we call the B-type elliptic Dunkl operator, also constitutes a mutually commuting family,

$$\left[\hat{y}_j^{\mathrm{B}}, \hat{y}_k^{\mathrm{B}}\right] = 0 \, . \tag{3.27}$$

Operators $\hat{y}_j^{\mathrm{B}}$ includes 'reflection terms' $\hat{t}_j\hat{t}_k\hat{s}_{jk}$ which follow from the boundary K-operators. From the B-type differential-difference operators which commute each other, we can construct a set of 'Hamiltonians' of the quantum many-body problem as

$$\hat{I}_n^{\mathrm{B}} = \sum_{j=1}^{N} \left(\hat{y}_j^{\mathrm{B}}\right)^n . \tag{3.28}$$

As the first non-trivial operators, one obtains

$$\begin{aligned}\hat{I}_2^{\mathrm{B}} = \sum_j \partial_j^2 &+ \kappa_1^2 \sum_{j,k}{}' \{-\wp(z_{jk}) - \wp(z_j + z_k) + \wp(\xi_{jk}) + \wp(\xi_j + \xi_k)\} \\ &+\kappa_1 \sum_{j,k}{}' \{\sigma'_{\xi_{jk}}(z_{jk})\hat{s}_{jk} + \sigma'_{\xi_j+\xi_k}(z_j + z_k)\hat{t}_j\hat{t}_k\hat{s}_{jk}\} \\ &+\kappa_2^2 \sum_j \{-\wp(z_j) + \wp(2\xi_j)\} + \kappa_2 \sum_j \sigma'_{2\xi_j}(z_j)\hat{t}_j \, .\end{aligned}$$

We see that coefficients for ξ^{-2} do not depend on dynamical variables $\{z_j\}$. By subtracting terms of $\mathrm{O}(\xi^{-2})$ and taking all rapidities to zero ($\xi \to 0$), we find that operator $\hat{I}_2^{\mathrm{B}}$ has the form

$$\begin{aligned}\hat{I}_2^{\mathrm{B}} \to \sum_{j=1}^{N} \partial_j^2 &- \sum_{j,k=1}^{N}{}' \{\wp(z_{jk}) \cdot \left(\kappa_1 \hat{P}_{jk} + \kappa_1^2\right) + \wp(z_j + z_k) \cdot \left(\kappa_1 \hat{Q}_j \hat{Q}_k \hat{P}_{jk} + \kappa_1^2\right)\} \\ &- \sum_{j=1}^{N} \wp(z_j) \cdot \left(\kappa_2 \hat{Q}_j + \kappa_2^2\right)\end{aligned} \tag{3.29}$$

which coincides with the Hamiltonian of the B-type elliptic CSM model with $su(n)$ spin [26]. Notice that in this case we restrict the functional space to

$$\hat{P}_{jk}\hat{s}_{jk} = 1 \qquad \hat{Q}_j\hat{t}_j = 1 \tag{3.30}$$

where operator $\hat{Q}_j$ acts on a spin space of jth particle, and satisfies

$$\hat{Q}_j^2 = 1 \qquad \hat{P}_{jk}\hat{Q}_j = \hat{Q}_k\hat{P}_{jk} \, .$$

As the second example, we choose K^{II} and $\bar{K}^{\mathrm{II}}$ boundary operators in the qKZ-type difference operator $\hat{Y}_j$ (equation (3.24)). By the same calculations, we get a set of the mutually commuting differential-difference operator,

$$\left[\hat{y}_j^{\mathrm{C}}, \hat{y}_k^{\mathrm{C}}\right] = 0 \tag{3.31}$$

in which the differential-difference operator $\hat{y}_j^{\mathrm{C}}$ is calculated as

$$\hat{y}_j^{\mathrm{C}} = \partial_j + \kappa_1 \sum_{k:k\neq j}^{N} \{\sigma_{\xi_{jk}}(z_{jk})\hat{s}_{jk} + \sigma_{\xi_j+\xi_k}(z_j+z_k)\hat{t}_j\hat{t}_k\hat{s}_{jk}\} + \kappa_3\sigma_{\xi_j}(2z_j)\hat{t}_j\,. \tag{3.32}$$

In this case, we have used the gauge-transformation by a C-type product function

$$H_{a,b}^{\mathrm{C}}(z) = \prod_{j<k}^{N}(\vartheta_1(z_{jk})\vartheta_1(z_j+z_k))^a \cdot \prod_{j=1}^{N}(\vartheta_1(2z_j))^b\,.$$

See the difference in operators $\hat{y}_j^{\mathrm{B}}$ and $\hat{y}_j^{\mathrm{C}}$; in coefficient of $\hat{t}_j$, the rapidity ξ and functional variable z play opposite roles. In this case the Hamiltonian can be calculated as

$$\begin{aligned}\hat{I}_2^{\mathrm{C}} &= \sum_{j=1}^{N}\left(\hat{y}_j^{\mathrm{C}}\right)^2\\ &\to \sum_{j=1}^{N}\partial_j^2 - \sum_{j,k=1}^{N}{}'\{\wp(z_{jk})\cdot\left(\kappa_1\hat{P}_{jk}+\kappa_1^2\right) + \wp(z_j+z_k)\cdot\left(\kappa_1\hat{Q}_j\hat{Q}_k\hat{P}_{jk}+\kappa_1^2\right)\}\\ &\quad - \sum_{j=1}^{N}\wp(2z_j)\cdot\left(2\kappa_3\hat{Q}_j+\kappa_3^2\right)\end{aligned} \tag{3.33}$$

which is called the Hamiltonian of the C-type CSM model [26].

As the last case, let us choose K^{I}- and $\bar{K}^{\mathrm{II}}$-operators (or, K^{II}- and $\bar{K}^{\mathrm{I}}$-operators) in the qKZ operator $\hat{Y}_j$. In the same manner, using the gauge-transformations by a BC-type elliptic product function

$$H_{a,b,c}^{\mathrm{BC}}(z) = \prod_{j<k}^{N}(\vartheta_1(z_{jk})\vartheta_1(z_j+z_k))^a \cdot \prod_{j=1}^{N}(\vartheta_1(2z_j))^b(\vartheta_1(z_j))^c \tag{3.34}$$

we obtain the BC-type Dunkl operator,

$$\hat{y}_j^{\mathrm{BC}} = \partial_j + \kappa_1 \sum_{k:k\neq j}^{N}\{\sigma_{\xi_{jk}}(z_{jk})\hat{s}_{jk} + \sigma_{\xi_j+\xi_k}(z_j+z_k)\hat{t}_j\hat{t}_k\hat{s}_{jk}\} + (\kappa_2\sigma_{2\xi_j}(z_j) + \kappa_3\sigma_{\xi_j}(2z_j))\hat{t}_j \tag{3.35}$$

and the Hamiltonian is calculated from $\hat{I}_2^{\mathrm{BC}}$;

$$\begin{aligned}\hat{I}_2^{\mathrm{BC}} &= \sum_{j=1}^{N}\left(\hat{y}_j^{\mathrm{BC}}\right)^2\\ &= \sum_j \partial_j^2 + \kappa_1 \sum_{j,k}{}'\{\sigma'_{\xi_{jk}}(z_{jk})\hat{s}_{jk} + \sigma'_{\xi_j+\xi_k}(z_j+z_k)\hat{t}_j\hat{t}_k\hat{s}_{jk}\}\\ &\quad + \kappa_1^2\sum_{j,k}{}'\{-\wp(z_{jk}) + \wp(\xi_{jk}) - \wp(z_j+z_k) + \wp(\xi_j+\xi_k)\}\\ &\quad + \sum_j\{\kappa_2\sigma'_{2\xi_j}(z_j) + 2\kappa_3\sigma'_{\xi_j}(2z_j)\}\hat{t}_j\\ &\quad + \sum_j\{\kappa_2^2(-\wp(z_j)+\wp(2\xi_j)) + \kappa_3^2(-\wp(2z_j)+\wp(\xi_j)) + \kappa_2\kappa_3(-\wp(z_j)+\wp(\xi_j))\}\,.\end{aligned}$$

In limiting case, $\xi \to 0$, the Hamiltonian reduces to the following form:

$$\hat{I}_2^{\mathrm{BC}} \to \sum_{j=1}^{N} \partial_j^2 - \sum_{j,k=1}^{N}{}' \{\wp(z_{jk}) \cdot (\kappa_1 \hat{P}_{jk} + \kappa_1^2) + \wp(z_j + z_k) \cdot (\kappa_1 \hat{Q}_j \hat{Q}_k \hat{P}_{jk} + \kappa_1^2)\}$$
$$- \sum_{j=1}^{N} \{\wp(z_j) \cdot (\kappa_2 \hat{Q}_j + \kappa_2^2 + \kappa_2\kappa_3) + \wp(2z_j) \cdot (2\kappa_3 \hat{Q}_j + \kappa_3^2)\} \tag{3.36}$$

which should be regarded as the Hamiltonian of the BC-type elliptic CSM model.

Notice that the Hamiltonian of the D-type CSM model corresponds to the B- or C-type CSM model without external potential [26],

$$\hat{I}_2^{\mathrm{D}} = \sum_{j=1}^{N} \partial_j^2 - \sum_{j,k=1}^{N}{}' \{\wp(z_{jk}) \cdot \left(\kappa_1 \hat{P}_{jk} + \kappa_1^2\right) + \wp(z_j + z_k) \cdot \left(\kappa_1 \hat{Q}_j \hat{Q}_k \hat{P}_{jk} + \kappa_1^2\right)\} . \tag{3.37}$$

This operator can be constructed from the D-type Dunkl operator,

$$\hat{y}_j^{\mathrm{D}} = \partial_j + \kappa_1 \sum_{k:k\neq j}^{N} \{\sigma_{\xi_{jk}}(z_{jk}) \hat{s}_{jk} + \sigma_{\xi_j+\xi_k}(z_j + z_k) \hat{t}_j \hat{t}_k \hat{s}_{jk}\} \tag{3.38}$$

which constitute a mutually commuting family,

$$\left[\hat{y}_j^{\mathrm{D}}, \hat{y}_k^{\mathrm{D}}\right] = 0 . \tag{3.39}$$

4. Concluding remarks

In this paper we have studied the infinite-dimensional representation of the R- and K-matrices, which satisfy both the Yang–Baxter equation and reflection equation. Following the idea of [2, 3], we regard R and K as elliptic operators acting on the functional space. Actually by restricting the functional space to the finite-dimensional space, the Belavin's elliptic R-matrix and its associated boundary K-matrix can be computed [9, 10]. As degenerate case we have studied the algebraic structure of the trigonometric R- and K-matrices, and obtain the defining relations for constant R- and K-matrices. As both R- and K-matrices represent the Hecke algebra, RE should be viewed as the interrelation between two representations [7].

In the second part, we derived the explicit forms of the elliptic Dunkl operator associated with the classical root systems. We have shown that the Dunkl operators are in fact obtainable as the quasi-classical limit of the quantum Knizhnik–Zamolodchikov-type difference operators. As a consequence, it is easy to see that the elliptic Dunkl operator constitutes a family of the mutually commuting differential-difference operators. From the mutually commutativity, one can define a set of Hamiltonians of the quantum many-body problems in terms of the Dunkl operators. One of the higher integrable operators coincides with the Hamiltonian of the elliptic CSM model associated with the classical root systems [26].

It has appeared in recent works [28] that the trigonometric Dunkl operators are closely related with the Macdonald polynomials and the character formula in conformal field theory. Also known is that the affine character for the level-1 su(n) WZNW model can be regarded as the large-N limit of the Rogers–Szegö polynomial, which is a certain limit of the Macdonald polynomial [15, 29]. The Dunkl operator formalism may help us to generalize these facts.

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